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# Certain Dual Integral Equations and Sonine's Integrals

A. S. PETERS

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# Certain Dual Integral Equations and Sonine's Integrals

## 1. Introduction

An examination of Sneddon's book [1] shows that a considerable number of problems in mathematical physics can be reduced to the solution of the simultaneous equations

$$(A) \quad \begin{aligned} \int_0^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi &= f(x) & 0 < x < 1 \\ \int_0^{\infty} J_{\mu}(\xi x) \phi(\xi) d\xi &= 0 & 1 < x < \infty \end{aligned}$$

which are called "dual" integral equations. These equations were solved for the case  $\omega > 0$  by Titchmarsh [2] who used Mellin transforms coupled with function theory techniques. Titchmarsh's work was extended by Busbridge [3] who solved the equations subject to the restrictions  $-2 < \omega$ ;  $-\mu-1 < \omega - \frac{1}{2} < \mu + 1$ . The equations (A) or related equations have also been formally solved by Tranter [4]; Gordon [5]; Ahiezer [6]; Noble [7]; and Sneddon [8]. The last four authors base their analyses on the special properties of the Bessel function kernels. At one point or other in the works of Gordon, Ahiezer and Sneddon, certain discontinuous integrals, such as Weber's, from the theory of Bessel functions, are used to find the solution. It is our impression that the full power of a method based on the use of Sonine's discontinuous integrals has



not been applied. The method which we develop here is different from those used earlier; but a part of it can be regarded as an amplification and extension of Gordon's work.

In section 3 we show how the equations

$$(B) \quad \begin{aligned} \int_0^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi &= f(x) & 0 < x < 1 \\ \int_0^{\infty} J_{\nu}(\xi x) \phi(\xi) d\xi &= g(x) & 1 < x < \infty \end{aligned}$$

can be solved in a rather simple way by the systematic application of Sonine's integrals. We show how Busbridge's restrictions can be relaxed, and we find a formula for the solution of (B) which is more general than the one which has hitherto been used. The solution formula is presented in various forms.

In section 4 we show that the method can be used to solve the more general dual equations

$$(C) \quad \begin{aligned} \int_k^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi &= f(x) & 0 < x < 1 \\ \int_0^{\infty} (\xi^2 - k^2)^{\alpha} J_{\nu}(\xi x) \phi(\xi) d\xi &= g(x) & 1 < x < \infty \end{aligned}$$

and in section 5 we show that certain other integral equations yield to the application of Sonine's integrals.

The equations (B) can be solved by using Mellin transforms to reduce (B) to a barrier\* equation which can be solved with

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\* We use this word to denote certain equations involving limits, which arise in connection with the solution of Hilbert's problem. [See Singular Integral Equations by N.I. Muskhelishvili.]





the aid of Plemelj's formulas. If one is familiar with Sonine's integrals, however, barrier equation techniques, or Wiener-Hopf techniques for the solution of (B) are more involved than the operational technique based on Sonine's integrals. Also, one of the points of the following is that equations (C) and other related ones are not in a form suitable for the direct application of transforms, and subsequent barrier equation techniques. The equations could be prepared for the application of transforms by using Sonine's integrals but after preparation it is simpler to continue with the method of discontinuous integrals.



## 2. Sonine's Integrals.

N.J. Sonine [Math. Ann. XVI (1880)] showed that

$$(2.1) \quad \int_0^{\pi/2} J_{\sigma}(Z \cos \theta) \cos^{\sigma+1} \theta J_{\mu}(z \sin \theta) \sin^{\mu+1} \theta d\theta \\ = \frac{z^{\mu} Z^{\sigma} J_{\mu+\sigma+1}(\sqrt{Z^2+z^2})}{(\sqrt{Z^2+z^2})^{\mu+\sigma+1}}$$

provided each of the real parts of  $\sigma$  and  $\mu$  exceeds minus one.

A discussion of this formula and some of its implications can be found in Watson's treatise on the Theory of Bessel Functions page 376. The substitutions  $Z = \beta \xi$ ;  $z = \alpha \xi$ ;  $\xi \sin \theta = t$ ;  $\sigma = s - \mu - 1$  change Sonine's integral into the form

$$(2.2) \quad \int_0^{\xi} \left[ J_{s-\mu-1}(\beta \sqrt{\xi^2-t^2}) \right] (\sqrt{\xi^2-t^2})^{s-\mu-1} J_{\mu}(\alpha t) t^{\mu+1} dt \\ = \frac{\alpha^{\mu} \beta^{s-\mu-1} \xi^s J_s(\xi \sqrt{\alpha^2+\beta^2})}{(\sqrt{\alpha^2+\beta^2})^s}$$

$$R_x s > R_y \mu > -1.$$

If we recall that

$$\lim_{z \rightarrow 0} \frac{J_{\lambda}(z)}{z^{\lambda}} = \frac{1}{2^{\lambda} \Gamma(\lambda+1)}$$

we see that we can divide each side of (2.2) by  $\beta^{s-\mu-1}$  and take the limit as  $\beta \rightarrow 0$ . This gives



$$(2.3) \quad \int_0^{\xi} (\xi^2 - t^2)^{s-\mu-1} J_{\mu}(\alpha t) t^{\mu+1} dt = 2^{s-\mu-1} \Gamma(s-\mu) \frac{\xi^s J_s(\alpha \xi)}{\alpha^{s-\mu}}$$

$$R_{\xi} s > R_{\xi} \mu > -1.$$

The integral on the left of (2.3) may be regarded as an operation which increases the order of  $J_{\mu}(\alpha t)$  to  $s$ .

In the paper mentioned above Sonine also investigated the infinite discontinuous integral (Watson p. 415)

$$(2.4) \quad \int_0^{\infty} J_p(\beta t) \frac{J_v(\alpha \sqrt{t^2 + z^2})}{(\sqrt{t^2 + z^2})^v} t^{p+1} dt = \begin{cases} 0 & 0 < \alpha < \beta \\ \frac{\beta^p}{\alpha^v} \left( \frac{\sqrt{\alpha^2 - \beta^2}}{z} \right)^{v-p-1} J_{v-p-1} \left( \frac{z}{\sqrt{\alpha^2 - \beta^2}} \right) & 0 < \beta < \alpha \end{cases}$$

$$z \text{ unrestricted; } R_{\xi} v > R_{\xi} p > -1.$$

The substitutions  $t = \sqrt{\tau^2 - z^2}$ ;  $p = v - 1 - r$  change this integral into

$$(2.5) \quad \int_z^{\infty} J_{v-1-r}(\beta \sqrt{\tau^2 - z^2}) (\sqrt{\tau^2 - z^2})^{v-1-r} \frac{J_v(\alpha \tau)}{\tau^{v-1}} d\tau = \begin{cases} 0 & 0 < \alpha < \beta \\ \frac{\beta}{\alpha^v} \left( \frac{\sqrt{\alpha^2 - \beta^2}}{z} \right)^r J_r \left( \frac{z}{\sqrt{\alpha^2 - \beta^2}} \right) & 0 < \beta < \alpha \end{cases}$$

$$z \text{ unrestricted; } R_{\xi} v > R_{\xi} r > -1.$$

If we divide (2.5) by  $\beta^{v-1-r}$  and then let  $\beta \rightarrow 0$  we find



$$(2.6) \quad \int_z^\infty (\tau^2 - z^2)^{v-1-r} \frac{J_v(\alpha\tau) d\tau}{\tau^{v-1}} = 2^{v-1-r} \Gamma(v-r) \frac{\alpha^{r-v}}{z^r} J_r(\alpha z)$$

provided

$$\alpha > 0 \quad \text{and} \quad R_\chi v > R_\chi r > R\left(\frac{v}{2} - \frac{3}{4}\right).$$

Here the operation presented by the integral on the left of (2.6) decreases the order of  $J_v(\alpha\tau)$  to  $r$ .

Another integral we will need, namely

$$(2.7) \quad \int_0^\infty J_p(\beta t) J_v(\alpha t) t^{p-v+1} dt = \begin{cases} 0 & 0 < \alpha < \beta \\ \frac{\beta^p (\alpha^2 - \beta^2)^{v-p-1}}{\alpha^v 2^{v-p-1} \Gamma(v-p)} & 0 < \beta < \alpha \end{cases}$$

$$R_\chi v > R_\chi p > -1$$

can be deduced from (2.4) by letting  $z \rightarrow 0$ .





### 3. Application of Sonine's Integrals to Certain Dual Integral Equations.

We proceed to show how Sonine's integrals can be used to find the solution of the dual equations

$$(3.1) \quad \int_0^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(3.2) \quad \int_0^{\infty} J_{\nu}(\xi x) \phi(\xi) d\xi = g(x) \quad 1 < x < \infty$$

for which we assume  $\mu, \nu$  each real, and  $\mu > -1; \nu > -1$ .

The basic idea is to use Sonine's integrals to modify the orders of the Bessel functions in (3.1) and (3.2) in such a way that we obtain two equations having the common kernel  $\xi^{k_{J_{\ell}}} J_{\ell}(x\xi)$ . In other words, we aim to reduce the equations to a Bessel transform

$$\int_0^{\infty} \xi^{k_{J_{\ell}}} J_{\ell}(x\xi) \phi(\xi) d\xi = \begin{cases} f_1(x) & 0 < x < 1 \\ g_1(x) & 1 < x < \infty \end{cases}$$

from which  $\phi(\xi)$  can easily be found by inversion.

Consider first the function

$$\phi_0(\xi) = \xi \int_1^{\infty} J_{\nu}(\xi t) t g(t) dt,$$

a function which satisfies



$$\int_0^{\infty} J_{\nu}(\xi x) \phi_0(\xi) d\xi = 0 \quad 0 < x < 1$$

$$\int_0^{\infty} J_{\nu}(\xi x) \phi_0(\xi) d\xi = g(x) \quad 1 < x < \infty.$$

If we write  $\phi(\xi) = \phi_0(\xi) + \psi(\xi)$ , equations (3.1), (3.2) reduce to

$$\int_0^{\infty} \xi^{\omega} J_{\mu}(\xi x) \psi(\xi) d\xi = f(x) - \int_0^{\infty} \xi^{\omega+1} J_{\mu}(\xi x) \int_1^{\infty} J_{\nu}(\xi t) t g(t) dt d\xi \quad 0 < x < 1$$

$$\int_0^{\infty} J_{\nu}(\xi x) \psi(\xi) d\xi = 0 \quad 1 < x < \infty.$$

This shows that it is sufficient to consider (3.1), (3.2) with  $g(x) = 0$ . We prefer, however, not to make use of this reduction because the method of solution we are going to apply directly to (3.1), (3.2) leads to a simple solution form which is more troublesome to obtain if the equations are first reduced as just described.

Our procedure is to be formal at first, and more rigorous later. The equation (3.1) can be converted into one involving a Bessel function of different order by multiplying each side of (3.1) by  $(\xi^2 - x^2)^{s-\mu-1} x^{\mu+1}$  and then integrating from 0 to  $\xi$ . If we assume that the order of integration can be changed in the resulting double integral and use (2.3) we find

$$(3.3) \int_0^{\infty} \xi^{\omega-s+\mu} \xi^s J_s(\xi \xi) \phi(\xi) d\xi = \frac{1}{2^{s-\mu-1} \Gamma(s-\mu)} \int_0^{\xi} (\xi^2 - x^2)^{s-\mu-1} f(x) x^{\mu+1} dx$$

for

$$s > \mu > -1 ; \quad 0 < \xi < 1.$$



Similarly, the equation (3.2) can be converted into one involving a Bessel function of order  $r$  by multiplying each side of (3.2) by  $(x^2 - \xi^2)^{v-1-r} x^{1-v}$  and then integrating from  $\xi$  to  $\infty$ .

If we do this and use (2.6) we find

$$(3.4) \quad \int_0^\infty \xi^{r-v} \zeta^{-r} J_r(\xi \zeta) \phi(\xi) d\xi = \frac{1}{2^{v-1-r} \Gamma(v-r)} \int_\zeta^\infty (x^2 - \zeta^2)^{v-1-r} g(x) x^{1-v} dx$$

for

$$v > r > \frac{v}{2} - \frac{3}{4}; \quad 1 < \zeta < \infty.$$

Now if we choose

$$s = r = \lambda; \quad \omega - s + \mu = r - v,$$

that is

$$\lambda = \frac{\omega + \mu + v}{2} = s = r$$

we have

$$(3.5) \quad \int_0^\infty \xi^{\lambda-v} J_\lambda(\xi \xi) \phi(\xi) d\xi = \frac{\xi^{-\lambda}}{2^{\lambda-\mu-1} \Gamma(\lambda-\mu)} \int_0^\xi (\xi^2 - x^2)^{\lambda-\mu-1} f(x) x^{\mu+1} dx \quad 0 < \xi < 1$$

$$(3.6) \quad \int_0^\infty \xi^{\lambda-v} J_\lambda(\xi \xi) \phi(\xi) d\xi = \frac{\xi^\lambda}{2^{v-1-\lambda} \Gamma(v-\lambda)} \int_\xi^\infty (x^2 - \xi^2)^{v-1-\lambda} g(x) x^{1-v} dx \quad 1 < \xi < \infty$$

subject to



$$(3.7) \quad \lambda > \mu > -1 \quad \text{and} \quad \nu > \lambda > \frac{\nu}{2} - \frac{3}{4}.$$

The inversion of (3.5) and (3.6) by the Hankel inversion formula yields

$$(3.8) \quad \xi^{\lambda-\nu-1} \phi(\xi) = \frac{1}{2^{\lambda-\mu-1} \Gamma(\lambda-\mu)} \int_0^1 \xi^{-\lambda+1} J_\lambda(\xi \xi) \int_0^\xi (\xi^2 - x^2)^{\lambda+\mu-1} f(x) x^{\mu+1} dx d\xi \\ + \frac{1}{2^{\nu-1-\lambda} \Gamma(\nu-\lambda)} \int_1^\infty \xi^{\lambda+1} J_\lambda(\xi \xi) \int_\xi^\infty (x^2 - \xi^2)^{\nu-1-\lambda} g(x) x^{1-\nu} dx d\xi.$$

The result (3.8) has been obtained subject to the limitation on the range of  $\omega$  implied by (3.7). However, this limitation can be changed by multiplying (3.3), (3.4) by suitable powers of  $\xi$ ; integrating and or differentiating with the aid of the formulas

$$(3.9) \quad \frac{d}{d\xi} \xi^s J_s(\xi \xi) = \xi \xi^s J_{s-1}(\xi \xi)$$

$$(3.10) \quad \frac{d}{d\xi} \xi^{-r} J_r(\xi \xi) = -\xi \xi^{-r} J_{r+1}(\xi \xi)$$

and then choosing  $r$  and  $s$  so that the kernels of the prepared equations are equal.

Let us consider for example, and in more detail, the equations

$$(3.11) \quad \int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(3.12) \quad \int_0^\infty J_\nu(\xi x) \phi(\xi) d\xi = 0 \quad 1 < x < \infty.$$





If we replace  $s$  in (3.3) by  $s+1$  we have

$$(3.13) \int_0^{\infty} \xi^{\omega-s+\mu-1} \zeta^{s+1} J_{s+1}(\xi\zeta) \phi(\xi) d\xi = \frac{1}{2^{s+\mu} \Gamma(s+\mu+1)} \int_0^{\xi} (\zeta^2 - x^2)^{s+\mu} f(x) x^{\mu+1} dx$$

$$(3.14) \int_0^{\infty} \xi^{r-v} \zeta^{-r} J_r(\xi\zeta) \phi(\xi) d\xi = 0$$

subject to  $s+1 > \mu > -1$  ;  $v > r > \frac{v}{2} - \frac{3}{4}$  . Assuming that it is permissible to differentiate under the integral sign in (3.14), the formula (3.9) shows that we can write (3.14) as

$$\frac{d}{d\xi} \int_0^{\infty} \xi^{r-v-1} \zeta^{r+1} J_{r+1}(\xi\zeta) \phi(\xi) d\xi = 0$$

and after integration

$$(3.141) \int_0^{\infty} \xi^{r-v-1} \zeta^{r+1} J_{r+1}(\xi\zeta) \phi(\xi) d\xi = c .$$

Now if we choose  $r = s = \lambda$  ;  $\omega - s + \mu - 1 = r - v - 1$  so that  $\lambda = \frac{\omega+\mu+v}{2}$  , as before, the equations (3.13) and (3.141) become

$$(3.15) \int_0^{\infty} \xi^{\lambda-v-1} J_{\lambda+1}(\xi\zeta) \phi(\xi) d\xi = \frac{\zeta^{-\lambda-1}}{2^{\lambda+\mu} \Gamma(\lambda+\mu+1)} \int_0^{\xi} (\zeta^2 - x^2)^{\lambda+\mu} f(x) x^{\mu+1} dx \quad 0 < \zeta < 1$$

$$(3.16) \int_0^{\infty} \xi^{\lambda-v-1} J_{\lambda+1}(\xi\zeta) \phi(\xi) d\xi = c \zeta^{-\lambda-1} \quad 1 < \zeta < \infty$$

subject to  $\lambda + 1 > \mu > -1$  ;  $v > \lambda > \frac{v}{2} - \frac{3}{4}$  which gives a range for  $\lambda$  different from the one given above. If the integral in



(3.15); (3.16) is to be continuous at  $\xi = 1$  we must take a special value for  $c$ , namely

$$\frac{1}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \int_0^1 (1-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx.$$

Because this is true, and because we do not wish to assume continuity we will write

$$c = \frac{1}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ \int_0^1 (1-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx + c_1 \right].$$

We can now find  $\phi(\xi)$  by applying Hankel's inversion formula to (3.15) and (3.16). First, the inverse of  $c\xi^{-\lambda-1}(\xi>1)$  is

$$c \int_1^\infty \xi^{-\lambda} J_{\lambda+1}(\xi\zeta) d\zeta = -\frac{c}{\xi} \int_1^\infty \frac{d[\zeta^{-\lambda} J_\lambda(\xi\zeta)] d\zeta}{d\zeta} = \frac{c}{\xi} J_\lambda(\xi)$$

for  $\lambda > -1/2$ . Hence, the inversion of (3.15); (3.16) yields

$$(3.17) \quad \phi(\xi) = \frac{\xi^{1-\lambda+\nu}}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ \begin{aligned} & c_1 J_\lambda(\xi) + J_\lambda(\xi) \int_0^1 (1-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx \\ & + \xi \int_0^1 \xi^{-\lambda} J_{\lambda+1}(\xi\zeta) \int_0^\zeta (\zeta^2-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx d\zeta \end{aligned} \right].$$

If  $f(x)$  is integrable over the interval from 0 to 1, the integrals in (3.17) exist for  $\lambda - \mu > -1$  or  $\omega > -2 + \mu - \nu$ . We can expect then that for this range of  $\omega$  the formula (3.17) gives in some sense the solution of (3.11) and (3.12).



The formula (3.17) can be presented in several forms. We can express it as

$$(3.18) \quad \phi(\xi) = \frac{\xi^{1-\lambda+\nu}}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ c_1 J_\lambda(\xi) + J_\lambda(\xi) \int_0^1 (1-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx - \int_0^1 \frac{d}{d\xi} [\xi^{-\lambda} J_\lambda(\xi \zeta)] \int_0^\zeta (\zeta^2 - x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx d\zeta \right]$$

and if

$$\lim_{\xi \rightarrow 0} \int_0^\xi (\zeta^2 - x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx = 0,$$

integration by parts gives

$$(3.19) \quad \phi(\xi) = \frac{\xi^{1-\lambda+\nu}}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ c_1 J_\lambda(\xi) + \int_0^1 \xi^{-\lambda} J_\lambda(\xi \zeta) \frac{d}{d\xi} \int_0^\zeta (\zeta^2 - x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx d\zeta \right]$$

Also if the Fourier-Bessel expansion

$$f(x) = \sum a_m J_\mu(\beta_m)$$

$$a_m = \frac{2}{J_{\mu+1}^2(\beta_m)} \int_0^1 x f(x) J_\mu(\beta_m x) dx ; \quad J_\mu(\beta_m) = 0$$

is substituted in (3.17) it becomes

$$\phi(\xi) = \xi^{1-\lambda+\nu} \left\{ \frac{c_1 J_\lambda(\xi)}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} + \sum \frac{a_m}{\beta_m^{\lambda-\mu+1}} \left[ J_\lambda(\xi) J_{\lambda+1}(\beta_m) + \xi \int_0^1 \zeta J_{\lambda+1}(\xi \zeta) J_{\lambda+1}(\beta_m \zeta) d\zeta \right] \right\}.$$



From

$$(3.20) \int \xi J_{\lambda+1}(\xi \zeta) J_{\lambda+1}(\beta_m \zeta) d\zeta = \zeta \left[ \frac{\xi J_{\lambda+2}(\xi \zeta) J_{\lambda+1}(\beta_m \zeta) - \beta_m J_{\lambda+1}(\xi \zeta) J_{\lambda+2}(\beta_m \zeta)}{\xi^2 - \beta_m^2} \right]$$

and the Bessel identities we find

$$(3.21) \phi(\xi) = \xi^{1-\lambda+\nu} \left\{ c_2 J_{\lambda}(\xi) + \sum \frac{a_m}{\beta_m^{\lambda-\mu}} \frac{[\xi J_{\lambda+1}(\xi) J_{\lambda}(\beta_m) - \beta_m J_{\lambda}(\xi) J_{\lambda+1}(\beta_m)]}{\xi^2 - \beta_m^2} \right\}.$$

For the case  $\mu = \nu$ , (3.17) (with  $c_1 = 0$ ) becomes

$$(3.22) \phi(\xi) = \frac{\xi^{1-\omega/2}}{2^{\omega/2} \Gamma(1+\frac{\omega}{2})} \left[ J_{\omega/2+\nu}(\xi) \int_0^1 (1-x^2)^{\omega/2} f(x) x^{\nu+1} dx + \xi \int_0^1 \xi^{-\omega/2-\nu} J_{\omega/2+\nu+1}(\xi \zeta) \int_0^1 (\zeta^2 - x^2)^{\omega/2} f(x) x^{\nu+1} dx d\zeta \right].$$

This is Busbridge's formula for the solution of the equations

$$\begin{aligned} \int_0^{\infty} \xi^{\omega} J_{\nu}(\xi x) \phi(\xi) d\xi &= f(x) & 0 < x < 1 \\ \int_0^{\infty} J_{\nu}(\xi x) \phi(\xi) d\xi &= 0 & 1 < x < \infty. \end{aligned}$$

Her analysis of these equations (an extension of Titchmarsh's work) depends upon the use of Mellin transforms, coupled with function theory procedures; and she found it necessary to impose the conditions

$$(3.23) \quad \omega > -2; \quad -\nu - 1 < \omega - \frac{1}{2} < \nu + 1.$$





We proceed to show that these restrictions can be relaxed if we regard the integrals in the dual equations from the point of view of summability. There is a practical reason for so regarding the integrals. In many problems of mathematical physics, the dual equations arise as a result of a limiting process such as

$$\lim_{z \rightarrow 0} \int_0^{\infty} \xi^{\omega} e^{-z\xi} J_{\mu}(\xi x) \phi(\xi) d\xi.$$

Hereafter in this section we will interpret the integrals in (3.1) and (3.2) as summable in a sense similar to the Cauchy sense. That is, we will interpret the integral in (3.1) to mean

$$\begin{aligned} (3.24) \quad \int_0^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi &= R \int_0^{\infty} \xi^{\omega} H_{\mu}^{(1)}(\xi x) \phi(\xi) d\xi \\ &= R \lim_{y \rightarrow 0} \int_0^{\infty} \xi^{\omega} H_{\mu}^{(1)}[\xi(x+iy)] \phi(\xi) d\xi, \quad y > 0 \\ &= -\frac{2}{\pi} R i e^{-\frac{\mu\pi i}{2}} \lim_{y \rightarrow 0} \int_0^{\infty} \xi^{\omega} K_{\mu}[\xi(y-ix)] \phi(\xi) d\xi \\ &= -\frac{2}{\pi} R i e^{-\frac{\mu\pi i}{2}} \lim_{y \rightarrow 0} \int_0^{\infty} \xi^{\omega} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi \end{aligned}$$

where  $R$  denotes the real part,  $K_{\mu}$  denotes the Bessel function of the third kind and  $z = x + iy$  with  $y > 0$ . Similarly, we will interpret the integral in (3.2) to mean

$$(3.25) \quad \int_0^{\infty} J_{\nu}(\xi x) \phi(\xi) d\xi = -\frac{2}{\pi} R i e^{-\frac{\nu\pi i}{2}} \lim_{y \rightarrow 0} \int_0^{\infty} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi.$$

Let us write (3.17) as



$$\phi(\xi) = \frac{\xi^{1-\lambda+\nu}}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ c J_{\lambda}(\xi) + \xi \int_0^1 \xi^{-\lambda} J_{\lambda+1}(\xi \zeta) F(\zeta) d\zeta \right]$$

where

$$F(\zeta) = \int_0^{\zeta} (\zeta^2 - x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx$$

subject to

$$\lambda = \frac{\omega+\mu+\nu}{2} ; \quad \lambda - \mu > -1.$$

Then

$$\int_0^{\infty} \xi^{\omega} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi = \frac{1}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ c \int_0^{\infty} \xi^{\lambda-\mu+1} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda}(\xi) d\xi + \int_0^1 \xi^{1-\lambda} F(\zeta) \int_0^{\infty} \xi^{\lambda-\mu+2} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda+1}(\xi \zeta) d\xi d\zeta \right].$$

The integrals on the right hand side of the last equation can be evaluated by using the formula

$$(3.26) \int_0^{\infty} \frac{K_p(at) J_q(bt) dt}{t^r} = \frac{b^q \Gamma\left(\frac{q-r+p+1}{2}\right) \Gamma\left(\frac{q-r-p+1}{2}\right)}{a^{q-r+1} 2^{r+1} \Gamma(q+1)} \cdot {}_2F_1\left(\frac{q-r+p+1}{2}, \frac{q-r-p+1}{2}; q+1; -\frac{b^2}{a^2}\right)$$

$$R_{\chi} a > |I b| ; \quad R_{\chi} (q+1-r) > |R_{\chi}(p)|$$

where  ${}_2F_1$  denotes, as usual, the hypergeometric function.



[See Watson, p. 410.] From this formula we find

$$(3.27) \int_0^{\infty} \xi^{\lambda-\mu+2} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda+1}(\xi \zeta) d\xi = \frac{e^{\frac{i\pi\mu}{2}} \zeta^{\lambda+1}}{z^{\mu}} (z^2 - \zeta^2)^{\mu-\lambda-2} 2^{\lambda+1+\mu} \Gamma(\lambda+\mu+2)$$

$$\operatorname{Im} z > 0; \quad \lambda > -2 + \frac{|\mu| + \mu}{2}$$

and

$$(3.28) \int_0^{\infty} \xi^{\lambda-\mu+1} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda}(\xi) d\xi = \frac{e^{\frac{i\pi\mu}{2}}}{z^{\mu}} (1-z^2)^{\mu-\lambda-1} 2^{\lambda-\mu} \Gamma(\lambda-\mu+1)$$

$$\operatorname{Im} z > 0; \quad \lambda > -1 + \frac{|\mu| + \mu}{2}.$$

Hence

$$\begin{aligned} \int_0^{\infty} \xi^{\omega} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi &= \frac{ce^{\frac{i\pi\mu}{2}}}{z^{\mu}} (1-z^2)^{\mu-\lambda-1} \\ &+ \frac{2(\lambda-\mu+1)e^{\frac{i\pi\mu}{2}}}{z^{\mu}} \int_0^1 \zeta (z^2 - \zeta^2)^{\mu-\lambda-2} F(\zeta) d\zeta \end{aligned}$$

which, since  $z$  is complex, is the same as

$$\begin{aligned} \int_0^{\infty} \xi^{\omega} K_{\mu}(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi &= ce^{\frac{i\pi\mu}{2}} (1-z^2)^{\mu-\lambda-1} \cdot \frac{1}{z^{\mu}} \\ &+ \frac{e^{\frac{i\pi\mu}{2}}}{z^{\mu+1}} \frac{d}{dx} \int_0^1 (\zeta^2 - z^2)^{\mu-\lambda-1} \zeta F(\zeta) d\zeta \\ &= \frac{ce^{\frac{i\pi\mu}{2}}}{z^{\mu}} (1-z^2)^{\mu-\lambda-1} \\ &+ \frac{(-1)^n e^{\frac{i\pi\mu}{2}} \Gamma(\mu-\lambda)}{2^n z^{\mu} \Gamma(\mu-\lambda+\eta)} \left( \frac{1}{z} \frac{\partial}{\partial x} \right)^{n+1} \int_0^1 (\zeta^2 - z^2)^{\mu-\lambda+\eta-1} \zeta F(\zeta) d\zeta \end{aligned}$$



where  $n$  is the least positive integer such that  $\mu - \lambda + n > 0$ .  
Under the interpretation (3.24) the last result leads to

$$\begin{aligned} \int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi &= -\frac{2}{\pi} R_\gamma \frac{ic(1-x^2)^{\mu-\lambda-1}}{x^\mu} \\ &- \frac{2}{\pi} R_\gamma \frac{i(-1)^n \Gamma(\mu-\lambda)}{2^n x^\mu \Gamma(\mu-\lambda+n)} \left( \frac{1}{x} \frac{\partial}{\partial x} \right)^{n+1} \int_0^1 (\xi^2 - x^2)^{\mu-\lambda+n-1} \xi F(\xi) d\xi \end{aligned}$$

which for  $x < 1$  is

$$\begin{aligned} \int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi &= \frac{2 \Gamma(\mu-\lambda) \sin \pi(\mu-\lambda)}{\pi x^\mu 2^n \Gamma(\mu-\lambda+n)} \left( \frac{1}{x} \frac{d}{dx} \right)^{n+1} \\ &\cdot \int_0^x (x^2 - \xi^2)^{\mu-\lambda+n-1} \xi \int_0^\xi (\xi^2 - t^2)^{\lambda-\mu} f(t) t^{\mu+1} dt d\xi. \end{aligned}$$

It is now easy to see (by applying the Laplace transform to the double integral of the last equation) that

$$\begin{aligned} \int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi &= \frac{2 \Gamma(\mu-\lambda) \sin \pi(\mu-\lambda) \Gamma(\lambda-\mu+1)}{\pi x^\mu 2^{n+1} n!} \left( \frac{1}{x} \frac{d}{dx} \right)^{n+1} \int_0^x (x^2 - t^2)^n f(t) t^{\mu+1} dt \\ &= \frac{1}{x^\mu 2^n n!} \left( \frac{1}{x} \frac{d}{dx} \right)^{n+1} \int_0^x (x^2 - t^2)^n f(t) t^{\mu+1} dt \\ &= f(x). \end{aligned}$$

Thus (3.17) satisfies the first equation (3.1).

For the second equation of the pair we have to evaluate





$$\int_0^{\infty} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi = \frac{1}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ c \int_0^{\infty} \xi^{1-\lambda+\nu} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda}(\xi) d\xi + \int_0^1 \xi^{-\lambda} F(\xi) \int_0^{\infty} \xi^{2-\lambda+\nu} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda+1}(\xi \xi) d\xi d\xi \right].$$

Using (3.26) again we have

$$(3.29) \quad \int_0^{\infty} \xi^{1-\lambda+\nu} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda}(\xi) d\xi = \left[ \xi^{-\lambda-1} \frac{d}{d\xi} \xi^{\lambda+1} \int_0^{\infty} \xi^{\nu-\lambda} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda+1}(\xi \xi) d\xi \right]_{\xi=1} \\ = \left[ \frac{-e^{\frac{i\pi\nu}{2}} z^{\nu} \Gamma(\nu+1)}{2^{\lambda-\nu+1} \Gamma(\lambda+1)} \xi^{-\lambda-1} \frac{d}{d\xi} \xi^{2\lambda+1} \int_0^1 (1-t)^{\lambda} (z^2 - t\xi^2)^{-\nu-1} dt \right]_{\xi=1}.$$

Also

$$(3.30) \quad \int_0^{\infty} \xi^{2-\lambda+\nu} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda+1}(\xi \xi) d\xi = \xi^{\lambda} \frac{d}{d\xi} \xi^{-\lambda} \int_0^{\infty} \xi^{\nu-\lambda+1} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda}(\xi \xi) d\xi \\ = \xi^{\lambda} \frac{d}{d\xi} \xi^{-2\lambda-1} \frac{d}{d\xi} \xi^{\lambda+1} \int_0^{\infty} \xi^{\nu-\lambda} K_{\nu}(\xi z e^{-\frac{i\pi}{2}}) J_{\lambda+1}(\xi \xi) d\xi \\ = \frac{-e^{\frac{i\pi\nu}{2}} z^{\nu} \Gamma(\nu+1)}{2^{\lambda-\nu+1} \Gamma(\lambda+1)} \xi^{\lambda} \frac{d}{d\xi} \xi^{-2\lambda-1} \frac{d}{d\xi} \xi^{2\lambda+1} \int_0^1 (1-t)^{\lambda} (z^2 - t\xi^2)^{-\nu-1} dt.$$

The integrals on the left hand sides of (3.29) and (3.30) converge if  $|z| > 0$  and if  $\nu+2 > |\nu|$ , a condition which is satisfied if  $\nu > -1$ . [The representation of these integrals in terms of integrals over  $(0,1)$ , as shown, is valid if  $\lambda > -1$ . However, it is clear that by expressing for example  $\int_0^{\infty} \xi^{\nu-\lambda+1} K_{\nu}(\xi z e^{-i\pi/2}) J_{\lambda}(\xi \xi) d\xi$  as higher derivatives, with respect to  $\xi$ , of a similar integral; the term  $(1-t)^{\lambda}$  in the representations can be made to appear



as  $(1-t)^{\lambda+n}$ .] Under the interpretation (3.25) the results (3.29) and (3.30) lead to

$$\int_0^\infty J_\nu(\xi x) \phi(\xi) d\xi = \frac{1}{\pi} R \frac{ix^\nu \Gamma(\nu+1)}{2^\omega \Gamma(\lambda-\mu+1) \Gamma(\lambda+1)} \left\{ c \left[ \frac{d}{d\xi} \xi^{2\lambda+1} \int_0^1 (1-t) (x^2 - t\xi^2)^{-\nu-1} dt \right]_{\xi=1} \right. \\ \left. + \int_0^1 F(\xi) \frac{d}{d\xi} \xi^{-2\lambda-1} \frac{d}{d\xi} \xi^{2\lambda+1} \int_0^1 (1-t) (x^2 - t\xi^2)^{-\nu-1} dt d\xi \right\}$$

but if  $x > 1$  the right hand side vanishes.

It should be observed that if  $\lambda = -n$  is a negative integer then the integrals in (3.27) and (3.28) exist simultaneously provided only  $2 > |\mu| + \mu$ . However, by virtue of the condition  $\lambda - \mu > -1$  which ensures the existence of  $F(\xi)$ ; and the condition  $\mu > -1$  we see that the only negative integer we can accept for  $\lambda$  is  $\lambda = -1$ ; and for this value we must have  $0 > \mu$ . For the case  $\lambda = -1$  the integrals (3.27) and (3.28) become

$$\int_0^\infty \xi^{1-\mu} K_\mu(\xi z e^{-\frac{i\pi}{2}}) J_0(\xi \xi) d\xi = \frac{\Gamma(1-\mu) e^{\frac{i\pi\mu}{2}} (\xi^2 - z^2)^{\mu-1}}{2^\mu z^\mu}$$

$$\int_0^\infty \xi^{-\mu} K_\mu(\xi z e^{-\frac{i\pi}{2}}) J_1(\xi) d\xi = \frac{\Gamma(1-\mu) e^{\frac{i\pi\mu}{2}} [(1-z^2)^\mu - z^{2\mu} e^{-i\pi\mu}]}{\mu 2^{\mu+1} z^\mu}$$

Therefore for  $\lambda = -1$ , and  $0 > \mu$ ,

$$\int_0^\infty \xi^\omega K_\mu(\xi z e^{-\frac{i\pi}{2}}) \phi(\xi) d\xi = \frac{1}{2^{-1-\mu} \Gamma(-\mu)} \left[ \frac{c \Gamma(1-\mu) e^{\frac{i\pi\mu}{2}}}{\mu 2^{\mu+1} z^\mu} \left\{ (1-z^2)^\mu - z^{2\mu} e^{-i\pi\mu} \right\} \right. \\ \left. + \int_0^1 \xi F(\xi) \frac{\Gamma(1-\mu) e^{\frac{i\pi\mu}{2}} (\xi^2 - z^2)^{\mu-1}}{2^\mu z^\mu} d\xi \right]$$

$$= - \frac{ce^{\frac{i\pi\mu}{2}}}{z^\mu} \left\{ (1-z^2)^\mu - z^{2\mu} e^{-i\pi\mu} \right\}$$



$$\begin{aligned}
& - \frac{2ie^{\frac{i\pi\mu}{2}}}{z^\mu} \int_0^1 \zeta F(\zeta) (\zeta^2 - z^2)^{\mu-1} d\zeta \\
& = - \frac{ce^{\frac{i\pi\mu}{2}}}{z^\mu} \left\{ (1-z^2)^\mu - z^{2\mu} e^{-i\pi\mu} \right\} \\
& + \frac{e^{\frac{i\pi\mu}{2}}}{z^{\mu+1}} \frac{\partial}{\partial x} \int_0^1 \zeta F(\zeta) (\zeta^2 - z^2)^\mu d\zeta
\end{aligned}$$

Hence

$$\int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi = - \frac{2}{\pi} \operatorname{Re} i \left\{ \frac{c}{x^\mu} [(1-x^2)^\mu - x^{2\mu} e^{-i\pi\mu}] + \frac{1}{x^{\mu+1}} \frac{\partial}{\partial x} \int_0^1 \zeta F(\zeta) (\zeta^2 - x^2)^\mu d\zeta \right\}.$$

For  $x < 1$ , this is

$$\int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi = \frac{2c}{\pi} x^\mu \sin \pi \mu + f(x)$$

which shows that we must take  $c = 0$ , i.e.,

$$c_1 = - \int_0^1 (1-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx$$

if (3.17) is to satisfy the equation (3.1) for the case  $\lambda = -1$ .

If  $\lambda = -1$  the integral (3.30) becomes

$$\begin{aligned}
\int_0^\infty \xi^{3+\nu} K_\nu \left( \xi z e^{-\frac{i\pi}{2}} \right) J_0(\xi \zeta) d\xi &= \frac{1}{\zeta} \frac{\partial}{\partial \zeta} \zeta \int_0^\infty \xi^{2+\nu} K_\nu \left( \xi z e^{-\frac{i\pi}{2}} \right) J_1(\xi \zeta) d\xi \\
&= \frac{2^{\nu+1} z^\nu e^{\frac{i\pi\nu}{2}} \Gamma(2+\nu)}{\zeta} \frac{\partial}{\partial \zeta} \frac{\zeta^2}{(z^2 - \zeta^2)^{2+\nu}}
\end{aligned}$$



from which

$$\int_0^{\infty} J_v(\xi x) \phi(\xi) d\xi = - \frac{2}{\pi 2^{-1-\mu} \Gamma(-\mu)} R_{\gamma} i^{2^{v+1} x^v} \Gamma(2+v) \int_0^1 F(\xi) \frac{\partial}{\partial \xi} \left[ \frac{\xi^2}{(x^2 - \xi^2)^{2+v}} \right] d\xi.$$

Since the right hand side of the last equation vanishes when  $x > 1$  we see that (3.2) is satisfied.

We have now shown that if  $f(x)$  is integrable, the function

$$(3.31) \quad \phi(\xi) = \frac{\xi^{1-\lambda+v}}{2^{\lambda-\mu} \Gamma(\lambda-\mu+1)} \left[ c_1 J_{\lambda}(\xi) + J_{\lambda}(\xi) \int_0^1 (1-x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx \right. \\ \left. + \xi \int_0^1 \xi^{-\lambda} J_{\lambda+1}(\xi \xi) \int_0^{\xi} (\xi^2 - x^2)^{\lambda-\mu} f(x) x^{\mu+1} dx d\xi \right]$$

$$\lambda = \frac{\omega + \mu + v}{2}$$

satisfies the dual integral equations

$$(3.32) \quad \int_0^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi = R_{\gamma} L \int_0^{\infty} \xi^{\omega} H_{\mu}^{(1)}(\xi z) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(3.33) \quad \int_0^{\infty} J_v(\xi x) \phi(\xi) d\xi = R_{\gamma} L \int_0^{\infty} H_v^{(1)}(\xi z) \phi(\xi) d\xi = 0 \quad 1 < x < \infty$$

where  $\mu > -1$ ;  $v > -1$ ; and  $Iz = I(x+iy) > 0$ ; if

$$(3.34) \quad \lambda > -1 + \frac{|\mu| + \mu}{2}$$

or in terms of  $\omega, \mu, v$ ; if





$$(3.35) \quad \omega > -2 + |\mu| - \nu.$$

If  $\lambda = -1$ , the function

$$(3.36) \quad \phi(\xi) = \frac{\xi^{3+\nu} 2^{1+\mu}}{\Gamma(-\mu)} \int_0^1 \xi J_0(\xi \xi) \int_0^\xi (\xi^2 - x^2)^{-1-\mu} f(x) x^{\mu+1} dx d\xi$$

satisfies the equations provided  $0 > \mu$ .

As one can expect, these results also hold for the interpretations

$$\begin{aligned} \int_0^\infty \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi &= \lim_{y \rightarrow 0+} \int_0^\infty e^{-y\xi} \xi^\omega J_\mu(\xi x) \phi(\xi) d\xi \\ \int_0^\infty J_\nu(\xi x) \phi(\xi) d\xi &= \lim_{y \rightarrow 0+} \int_0^\infty e^{-y\xi} J_\nu(\xi x) \phi(\xi) d\xi. \end{aligned}$$

The proof of this, however, is considerably more involved than the proof above.

The appearance of  $c$ , in (3.31) shows that the solution of the dual equations is not unique unless  $\phi(\xi)$  is subjected to additional restrictions at zero and infinity. Also other solutions can in fact be found if we allow  $f(x)$  to have say a delta function component at  $x = 1$ . We prefer to hold conditions for uniqueness in abeyance so that they may be appropriately fixed by the physical demands of a situation which leads to a consideration of the dual equations.

Recently, Fredricks [9] noted that the equations

$$(D) \quad \begin{aligned} \int_0^\infty \xi^{-1} Q(\xi) \sin \xi x d\xi &= h(x) & 0 \leq x < 1 \\ \int_0^\infty Q(\xi) \sin \xi x d\xi &= 0 & 1 < x < \infty \end{aligned}$$



and

$$(E) \quad \begin{aligned} \int_0^{\infty} \xi^{-1} P(\xi) \cos \xi x \, d\xi &= g(x) & 0 \leq x < 1 \\ \int_0^{\infty} P(\xi) \cos \xi x \, d\xi &= 0 & 1 < x < \infty \end{aligned}$$

are not covered by Busbridge's formula (3.22) because they do not satisfy Busbridge's conditions

$$(3.23) \quad \omega > -2; \quad -\nu - 1 < \omega - \frac{1}{2} < \nu + 1.$$

Fredericks solved these equations by first writing

$$\begin{aligned} h(x) &= \sum_{n=1}^{\infty} b_n \sin n \pi x & b_n &= 2 \int_0^1 h(x) \sin n \pi x \, dx \\ g(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \pi x & a_n &= 2 \int_0^1 g(x) \cos n \pi x \, dx. \end{aligned}$$

Then, using the expansions

$$(3.37) \quad \sin n \pi x = 2 \sum_{m=0}^{\infty} J_{2m+1}(n\pi) \sin[(2m+1)\arcsin x]$$

$$(3.38) \quad \cos n \pi x = J_0(n\pi) + 2 \sum_{m=1}^{\infty} J_{2m}(n\pi) \cos[2m \arcsin x]$$

he found the components of the solutions corresponding to  $\sin[(2m+1)\arcsin x]$  and  $\cos[2m \arcsin x]$  by using some of Weber's discontinuous integrals for example:



$$\int_0^{\infty} \xi^{-1} J_{2m}(\xi) \cos \xi x \, d\xi = \frac{\cos[2m \arcsin x]}{2m} \quad 0 \leq x < 1$$

$$\int_0^{\infty} J_{2m}(\xi) \cos \xi x \, d\xi = 0 \quad 1 < x < \infty$$

In this way Fredericks found the solutions in the form of doubly infinite series, namely

$$(3.39) \quad Q(\xi) = 2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_n (2m+1) J_{2m+1}(n\pi) J_{2m+1}(\xi)$$

$$(3.40) \quad P(\xi) = \xi J_1(\xi) \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n J_0(n\pi) \right\} + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n m J_{2m}(n\pi) J_{2m}(\xi)$$

which he showed are absolutely convergent. It perhaps should be remarked in passing that if  $g(x) = 1$  then  $a_0 = 2$ ;  $a_n = 0$   $n \neq 0$ ; and (3.40) gives  $P(\xi) = \xi J_1(\xi)$  but  $\int_0^{\infty} \xi J_1(\xi) \cos \xi x \, d\xi$  does not exist in the ordinary sense.

Although (D) and (E) are not covered by Busbridge's conditions they are subsumed by formulae we have derived above. The equations (E) can be written

$$\int_0^{\infty} \xi^{-1} \xi^{1/2} P(\xi) J_{-1/2}(\xi x) d\xi = \sqrt{\frac{2}{\pi}} \frac{g(x)}{\sqrt{x}} \quad 0 < x < 1$$

$$\int_0^{\infty} \xi^{1/2} P(\xi) J_{-1/2}(\xi x) d\xi = 0 \quad 1 < x < \infty$$

Here  $\omega = -1$ ;  $\mu = \nu = -1/2$ ;  $\lambda = \frac{\omega + \mu + \nu}{2} = -1$ . Hence the solution is given by (3.36), that is,



$$(3.41) \quad P(\xi) = \frac{2}{\pi} \xi^2 \int_0^1 \zeta J_0(\xi \zeta) \int_0^\zeta (\zeta^2 - x^2)^{-1/2} g(x) dx d\zeta$$

This can be expanded into

$$(3.42) \quad P(\xi) = \frac{2\xi^2}{\pi} \int_0^1 \zeta J_0(\xi \zeta) \int_0^\zeta (\zeta^2 - x^2)^{-1/2} \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \right\} dx d\zeta$$

$$= \frac{2\xi^2}{\pi} \left[ \frac{a_0 \pi J_1(\xi)}{4\xi} + \sum_{n=1}^{\infty} \frac{a_n \pi}{2} \int_0^1 \zeta J_0(\xi \zeta) J_0(n\pi \zeta) d\zeta \right]$$

and this, as we have already noted in (3.20) and (3.21), is the same as

$$P(\xi) = \frac{2\xi^2}{\pi} \left\{ \frac{a_0 \pi J_1(\xi)}{4\xi} + \frac{\pi}{2} \sum_{n=1}^{\infty} a_n \left[ \frac{\xi J_1(\xi) J_0(n\pi) - n\pi J_0(\xi) J_1(n\pi)}{\xi^2 - (n\pi)^2} \right] \right\}$$

another form of the solution which may be useful. Fredericks' form of the solution comes from (3.41) by passing to (3.42); substituting (3.38) in (3.42); and then evaluating

$$\frac{2\xi^2}{\pi} \int_0^1 \zeta J_0(\xi \zeta) \int_0^\zeta (\zeta^2 - x^2)^{-1/2} \cos[2m \arcsin x] dx d\zeta$$

which, as some analysis will show, is equal to  $2m J_{2m}(\xi)$ .

The equation (D) can be written

$$\int_0^\infty \xi^{-1} \xi^{1/2} Q(\xi) J_{1/2}(\xi x) d\xi = \sqrt{\frac{2}{\pi}} \frac{h(x)}{\sqrt{x}} \quad 0 < x < 1$$

$$\int_0^\infty \xi^{1/2} Q(\xi) J_{1/2}(\xi x) d\xi = 0 \quad 1 < x < \infty.$$





Here  $\omega = -1$ ;  $\mu = \nu = 1/2$ ;  $\lambda = \frac{\omega+\mu+\nu}{2} = 0$ . The solution is given by (3.31), namely,

$$(3.43) \quad Q(\xi) = \frac{2\xi}{\pi} \left[ \begin{aligned} & \frac{c_1 \sqrt{\pi}}{2} J_0(\xi) + \int_0^1 (1-x^2)^{-1/2} {}_x h(x) dx \\ & + \xi \int_0^1 J_1(\xi \rho) \int_0^\rho (\rho^2 - x^2)^{-1/2} {}_x h(x) dx d\rho \end{aligned} \right] .$$

If we take  $c_1 = 0$  and suppose that  $h(x)$  is such that

$$\lim_{\rho \rightarrow 0} \int_0^\rho (\rho^2 - x^2)^{-1/2} {}_x h(x) dx = 0$$

as we will, then the solution is

$$(3.44) \quad Q(\xi) = \frac{2\xi}{\pi} \int_0^1 J_0(\xi \rho) \frac{d}{d\rho} \int_0^\rho (\rho^2 - x^2)^{-1/2} {}_x h(x) dx d\rho .$$

Frederick's form comes from (3.44) by substituting

$$h(x) = 2 \sum_{n=1}^{\infty} b_n \sum_{m=0}^{\infty} J_{2m+1}(n\pi) \sin[(2m+1) \arcsin x]$$

and showing, as some analysis does, that

$$\begin{aligned} & \frac{2\xi}{\pi} \int_0^1 J_0(\xi \rho) \frac{d}{d\rho} \int_0^\rho (\rho^2 - x^2)^{-1/2} x \sin[(2m+1) \arcsin x] dx d\rho \\ & = (2m+1) J_{2m+1}(\xi) . \end{aligned}$$



#### 4. A More General Pair of Equations.

The dual equations

$$(4.1) \quad \int_k^{\infty} \xi^{\omega} J_{\mu}(\xi x) \phi(\xi) d\xi = f(x) \quad \begin{array}{l} 0 < x < 1 \\ 0 \leq k \end{array}$$

$$(4.2) \quad \int_0^{\infty} (\xi^2 - k^2)^{\alpha} J_{\nu}(\xi x) \phi(\xi) d\xi = g(x) \quad 1 < x < \infty$$

may be regarded as a generalization of the pair (3.1) and (3.2). Equations of the type (4.1), (4.2) arise, for example, in connection with the problem of diffraction of waves by a circular disc. These equations are not in a form suitable for the application of transform techniques; but they can be solved in some cases by a method very similar to that used in the previous section. We content ourselves here with the showing of the formal procedure. Under the interpretation of the integrals used in the previous section, a proof of the results we obtain would be analogous to that given in section 3.

Multiply both sides of (4.1) by  $J_{s-\mu-1}(b\sqrt{\rho^2-x^2})(\sqrt{\rho^2-x^2})^{s-\mu-1}x^{\mu+1}$  and integrate from 0 to  $\rho$ . If we use (2.3) we find

$$(4.3) \quad \int_k^{\infty} \xi^{\omega+\mu} b^{s-\mu-1} \rho^s \frac{J_s(\rho\sqrt{\xi^2+b^2})}{(\sqrt{\xi^2+b^2})^s} \phi(\xi) d\xi \\ = \int_0^{\rho} J_{s-\mu-1}(b\sqrt{\rho^2-x^2})(\sqrt{\rho^2-x^2})^{s-\mu-1} f(x) x^{\mu+1} dx$$



subject to  $s > \mu > -1$ . Then, if we set  $b = ik$  and use the substitution  $\sqrt{\xi^2 - k^2} = \zeta$ , (4.3) becomes

$$\begin{aligned}
 (4.4) \quad & \int_0^\infty \zeta^{1-s} s J_s(\rho \zeta) \left( \sqrt{\zeta^2 + k^2} \right)^{\omega + \mu - 1} \phi(\sqrt{\zeta^2 + k^2}) d\zeta \\
 &= \int_0^\rho J_{s-\mu-1} \left( ik \sqrt{\rho^2 - x^2} \right) \left( \frac{\sqrt{\rho^2 - x^2}}{ik} \right)^{s-\mu-1} f(x) x^{\mu+1} dx \\
 &= \int_0^\rho I_{s-\mu-1} \left( k \sqrt{\rho^2 - x^2} \right) \left( \frac{\sqrt{\rho^2 - x^2}}{k} \right)^{s-\mu-1} f(x) x^{\mu+1} dx
 \end{aligned}$$

for  $0 < \rho < 1$ .

Next, if we multiply both sides of (4.2) by

$J_{v-1-r} \left( k \sqrt{x^2 - \rho^2} \right) \left( \sqrt{x^2 - \rho^2} \right)^{v-1-r} x^{1-v}$ , integrate from  $\rho$  to  $\infty$  and use the formula (2.5), namely

$$\begin{aligned}
 & \int_\rho^\infty J_{v-1-r} \left( k \sqrt{x^2 - \rho^2} \right) \left( \sqrt{x^2 - \rho^2} \right)^{v-1-r} J_v(\xi x) x^{1-v} dx \\
 &= \begin{cases} 0 & 0 < \xi < k \\ \frac{k^{v-1-r}}{\xi^v} \left( \frac{\sqrt{\xi^2 - k^2}}{\rho} \right)^r J_r \left( \rho \sqrt{\xi^2 - k^2} \right) & 0 < k < \xi \end{cases}
 \end{aligned}$$

$$Rv > Rr > -1$$

we find

$$\begin{aligned}
 & \int_k^\infty (\xi^2 - k^2)^\alpha \phi(\xi) \frac{k^{v-1-r}}{\xi^v} \left( \frac{\sqrt{\xi^2 - k^2}}{\rho} \right)^r J_r \left( \rho \sqrt{\xi^2 - k^2} \right) d\xi \\
 &= \int_\rho^\infty J_{v-1-r} \left( k \sqrt{x^2 - \rho^2} \right) \left( \sqrt{x^2 - \rho^2} \right)^{v-1-r} g(x) x^{1-v} dx ;
 \end{aligned}$$



and after the substitution  $\sqrt{\xi^2 - k^2} = \zeta$ ,

$$(4.5) \quad \int_0^\infty \zeta^{2\alpha+r+1} \rho^{-r} J_r(\rho\zeta) (\sqrt{\zeta^2 + k^2})^{-\nu-1} \phi(\sqrt{\zeta^2 + k^2}) d\zeta \\ = \int_\rho^\infty J_{\nu-1-r} \left( k \sqrt{x^2 - \rho^2} \right) \left( \frac{\sqrt{x^2 - \rho^2}}{k} \right)^{\nu-1-r} g(x) x^{1-\nu} dx$$

for  $\rho > 1$ .

Now if we choose

$$r = s = \lambda; \quad 1-s = 2\alpha+r+1$$

that is,  $\lambda = -\alpha$ ; the equations (4.4) and (4.5) reduce to

$$(4.6) \quad \int_0^\infty \zeta^{1-\lambda} J_\lambda(\rho\zeta) (\sqrt{\zeta^2 + k^2})^{\omega+\mu-1} \phi(\sqrt{\zeta^2 + k^2}) d\zeta \\ = \rho^{-\lambda} \int_0^\rho I_{\lambda-\mu-1} \left( k \sqrt{\rho^2 - x^2} \right) \left( \frac{\sqrt{\rho^2 - x^2}}{k} \right)^{\lambda-\mu-1} f(x) x^{\mu+1} dx, \quad 0 < \rho < 1$$

$$(4.7) \quad \int_0^\infty \zeta^{1-\lambda} J_\lambda(\rho\zeta) (\sqrt{\zeta^2 + k^2})^{-\nu-1} \phi(\sqrt{\zeta^2 + k^2}) d\zeta \\ = \rho^\lambda \int_\rho^\infty J_{\nu-1-\lambda} \left( k \sqrt{x^2 - \rho^2} \right) \left( \frac{\sqrt{x^2 - \rho^2}}{k} \right)^{\nu-1-\lambda} g(x) x^{1-\nu} dx, \quad 1 < \rho < \infty$$

subject to  $\lambda > \mu > -1$  and  $\nu > \lambda > -1$ ; which limit  $\alpha$  to  $\nu > -\alpha > \mu$ .

[This range for  $\alpha$  can be changed by performing preliminary manipulations on (4.4) and (4.5) before making the orders of the Bessel functions equal, as we have described in section 3.]

Hence, if the condition

$$(4.8) \quad \omega + \mu = -\nu$$





is satisfied we can invert (4.6) and (4.7). Inversion yields

$$\begin{aligned}
 (4.9) \quad & \xi^{-\lambda} (\sqrt{\xi^2 + k^2})^{-\nu-1} \phi(\sqrt{\xi^2 + k^2}) \\
 &= \int_0^1 \rho^{-\lambda+1} J_\lambda(\rho\xi) \int_0^\rho I_{\lambda-\mu-1}(k\sqrt{\rho^2-x^2}) \left(\frac{\sqrt{\rho^2-x^2}}{k}\right)^{\lambda-\mu-1} f(x) x^{\mu+1} dx d\rho \\
 &+ \int_1^\infty \rho^{\lambda+1} J_\lambda(\rho\xi) \int_\rho^\infty J_{\nu-1-\lambda}(k\sqrt{x^2-\rho^2}) \left(\frac{\sqrt{x^2-\rho^2}}{k}\right)^{\nu-1-\lambda} g(x) x^{1-\nu} dx d\rho .
 \end{aligned}$$

[It should be observed that after multiplying (4.6) by suitable powers of  $\rho$  two differentiations with respect to  $\rho$  will raise the power of  $\xi$  in the integral by two without changing the order of the Bessel functions. This implies that the condition (4.8) can be relaxed; that is, after preparation, the equations (4.6) and (4.7) can be inverted if  $\omega+\mu+\nu = 2n$  where  $n = 0, 1, 2, \dots$ .]

Ahiezer [6] solved the equations

$$(4.10) \quad \int_k^\infty J_\mu(x\xi) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(4.11) \quad \int_0^\infty J_{-\mu}(x\xi) \phi(\xi) d\xi = 0 \quad 1 < x < \infty .$$

$$k \geq 0 ; \quad 0 < \mu^2 < 1$$

and

$$(4.12) \quad \int_k^\infty J_0(x\xi) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(4.13) \quad \int_0^\infty (\xi^2 - k^2)^\alpha J_0(x\xi) \phi(\xi) d\xi = 0 \quad 1 < x < \infty$$

$$k \geq 0 , \quad 0 < \alpha^2 < 1$$



which are special cases of (4.1); (4.2). If  $-1 < \mu < 0$  the solution of (4.10); (4.11) is given by (4.9).

In order to illustrate some of the parenthetical points made in the previous paragraphs, let us consider in more detail the equations

$$(4.14) \quad \int_k^\infty \xi^{2n} J_0(x\xi) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(4.15) \quad \int_0^\infty (\xi^2 - k^2)^\alpha J_0(x\xi) \phi(\xi) d\xi = 0 \quad 1 < x < \infty$$

where  $n$  is an integer. If we replace  $s$  in (4.4) by  $s+1$  we see from (4.4) and (4.5) that (4.14); (4.15) can be changed into

$$(4.16) \quad \int_0^\infty \xi^{-s} \rho^{s+1} J_{s+1}(\rho\xi) (\sqrt{\xi^2 + k^2})^{2n-1} \phi(\sqrt{\xi^2 + k^2}) d\xi \\ = \int_0^\rho I_s(k\sqrt{\rho^2 - x^2}) \left( \frac{\sqrt{\rho^2 - x^2}}{k} \right)^s f(x) x dx$$

$$(4.17) \quad \int_0^\infty \xi^{2\alpha+r+1} \rho^{-r} J_r(\rho\xi) (\sqrt{\xi^2 + k^2})^{-1} \phi(\sqrt{\xi^2 + k^2}) d\xi = 0$$

subject to  $s+1 > 0$  and  $0 > r > -1$ . Assuming that it is permissible to differentiate under the integral sign in (4.17), the formula (3.9) shows that we can write (4.17) as

$$\frac{d}{d\rho} \int_0^\infty \xi^{2\alpha+r} \rho^{r+1} J_{r+1}(\rho\xi) (\sqrt{\xi^2 + k^2})^{-1} \phi(\sqrt{\xi^2 + k^2}) d\xi = 0$$

and after integration as



$$(4.18) \quad \int_0^{\infty} \xi^{2\alpha+r} \rho^{r+1} J_{r+1}(\rho\xi) \left(\sqrt{\xi^2+k^2}\right)^{-1} \phi\left(\sqrt{\xi^2+k^2}\right) d\xi = c.$$

If  $n = 0$  we can pass directly to the inversion of (4.16) and (4.18). If  $n$  is not zero, say  $n = 1$ , we proceed as follows. Equation (4.18) in turn can be written

$$\frac{d}{d\rho} \int_0^{\infty} \xi^{2\alpha+r-1} \rho^{-r} J_r(\rho\xi) \left(\sqrt{\xi^2+k^2}\right)^{-1} \phi\left(\sqrt{\xi^2+k^2}\right) d\xi = -c\rho^{-2r-1}$$

and integrated to give

$$(4.19) \quad \int_0^{\infty} \xi^{2\alpha+r-1} J_r(\rho\xi) \left(\sqrt{\xi^2+k^2}\right)^{-1} \phi\left(\sqrt{\xi^2+k^2}\right) d\xi = \frac{c\rho^{-r}}{2r} + c_1\rho^r.$$

Continuation of the process leads to

$$(4.20) \quad \begin{aligned} \frac{d}{d\rho} \int_0^{\infty} \xi^{2\alpha+r-2} \rho^{r+1} J_{r+1}(\rho\xi) \left(\sqrt{\xi^2+k^2}\right)^{-1} \phi\left(\sqrt{\xi^2+k^2}\right) d\xi &= \frac{c\rho}{2r} + c_1\rho^{2r+1} \\ \int_0^{\infty} \xi^{2\alpha+r-2} \rho^{r+1} J_{r+1}(\rho\xi) \left(\sqrt{\xi^2+k^2}\right)^{-1} \phi\left(\sqrt{\xi^2+k^2}\right) d\xi \\ &= \frac{c\rho^2}{4r} + \frac{c_1\rho^{2r+2}}{2r+2} + c_2. \end{aligned}$$

Multiplying the last equation by  $k^2$  and adding to (4.18) gives

$$(4.21) \quad \begin{aligned} \int_0^{\infty} \xi^{2\alpha+r-2} \rho^{r+1} J_{r+1}(\rho\xi) \left(\sqrt{\xi^2+k^2}\right) \phi\left(\sqrt{\xi^2+k^2}\right) d\xi \\ = \frac{k^2 c\rho^2}{4r} + \frac{k^2 c_1\rho^{2r+2}}{2r+2} + c_2 k^2 + c. \end{aligned}$$

Now with  $n = 1$ , if we take



$$r = s = \lambda, \quad -s = 2\alpha + r - 2, \quad \lambda = 1 - \alpha$$

equations (4.16); (4.21) reduce to

$$(4.22) \quad \int_0^{\infty} \xi^{\alpha-1} J_{2-\alpha}(\rho\xi) \left( \sqrt{\xi^2 + k^2} \right) \phi \left( \sqrt{\xi^2 + k^2} \right) d\xi \\ = \rho^{\alpha-2} \int_0^{\rho} I_{1-\alpha} \left( k \sqrt{\rho^2 - x^2} \right) \left( \frac{\sqrt{\rho^2 - x^2}}{k} \right)^{1-\alpha} f(x) x dx \quad 0 < \rho < 1$$

$$(4.23) \quad \int_0^{\infty} \xi^{\alpha-1} J_{2-\alpha}(\rho\xi) \sqrt{\xi^2 + k^2} \phi \left( \sqrt{\xi^2 + k^2} \right) d\xi \\ = a_1 \rho^{\alpha} + a_2 \rho^{\alpha-2} + a_3 \rho^{2-\alpha} \quad 1 < \rho < \infty$$

$$2-\alpha > 0; \quad 0 > 1-\alpha > -1$$

which are ready for inversion.





## 5. Some Related Equations.

The equations discussed above do not exhaust the kinds that can be solved by the application of Sonine's discontinuous integrals. For example, the dual equations

$$(5.1) \quad \int_0^{\infty} \xi^{\omega} J_{\mu} \left( x \sqrt{\xi^2 + b^2} \right) \phi(\xi) d\xi = f(x) \quad 0 < x < 1$$

$$(5.2) \quad \int_0^{\infty} J_{-\mu} \left( x \sqrt{\xi^2 + b^2} \right) \phi(\xi) d\xi = g(x) \quad 1 < x < \infty$$

can be solved by the method of section 4; and there are others which yield to this method.

The equation

$$(5.3) \quad x^{\omega} \int_0^1 J_{\mu}(x\xi) \phi(\xi) d\xi + \int_1^{\infty} J_{\nu}(x\xi) \phi(\xi) d\xi = f(x) \quad 0 < x < \infty$$

where  $\mu > -1$ ;  $\nu > -1$  and

$$(5.4) \quad \mu > \nu + |\omega| ,$$

one of a type apparently different from those discussed before, is in fact equivalent to a pair of dual equations. However, the following method of solution is easier and more direct than exploiting the equivalence. Multiply both sides of (5.3) by  $x^{\nu-s+1} J_s(px)$  and integrate from zero to infinity to obtain



$$\begin{aligned}
 (5.5) \quad & \int_0^1 \phi(\xi) \int_0^\infty x^{\omega+v-s+1} J_s(\rho x) J_\mu(\xi x) dx d\xi \\
 & + \int_1^\infty \phi(\xi) \int_0^\infty x^{v-s+1} J_v(\xi x) J_s(\rho x) dx d\xi \\
 & = \int_0^\infty x^{v-s+1} J_s(\rho x) f(x) dx .
 \end{aligned}$$

Now

$$(5.6) \quad \int_0^\infty x^{v-s+1} J_v(\xi x) J_s(\rho x) dx = \begin{cases} 0 & 0 < \rho < \xi \\ \frac{\xi^v (\rho^2 - \xi^2)^{s-v-1}}{\rho^s 2^{s-v-1} \Gamma(s-v)} & 0 < \xi < \rho \end{cases}$$

if

$$(5.7) \quad s > v > -1 .$$

Therefore if  $\rho < 1$

$$(5.8) \quad \int_0^1 \phi(\xi) \int_0^\infty x^{\omega+v-s+1} J_s(\rho x) J_\mu(\xi x) dx d\xi = \int_0^\infty x^{v-s+1} J_s(\rho x) f(x) dx .$$

Again,

$$(5.9) \quad \int_0^\infty x^{\omega+v-s+1} J_s(\rho x) J_\mu(\xi x) dx = \begin{cases} 0 & 0 < \xi < \rho \\ \frac{\rho^s (\xi^2 - \rho^2)^{\mu-s-1}}{\xi^\mu 2^{\mu-s-1} \Gamma(\mu-s)} & 0 < \rho < \xi \end{cases}$$

provided

$$(5.10) \quad \mu > s > -1$$

and

$$(5.11) \quad \omega+v-s+1 = s-\mu+1 .$$



With

$$s = \lambda = \frac{\omega + \mu + \nu}{2}$$

the conditions (5.7) and (5.10) are satisfied by virtue of the assumption (5.4); and (5.8) reduces to

$$(5.12) \quad \int_{\rho}^1 (\xi^2 - \rho^2)^{\mu - \lambda - 1} \frac{\phi(\xi)}{\xi^{\mu}} d\xi \\ = \rho^{-\lambda} 2^{\mu - \lambda - 1} \Gamma(\mu - \lambda) \int_0^{\infty} x^{\nu - \lambda + 1} J_{\lambda}(\rho x) f(x) dx \quad 0 < \rho < 1.$$

The substitutions  $\xi^2 = 1 - z$ ;  $\rho^2 = 1 - \sigma$  reduce (5.12) to a generalized Abel equation from which  $\phi(\xi)$  can easily be found for the interval  $0 < \xi < 1$ .

In order to find  $\phi(\xi)$  for  $1 < \xi < \infty$  consider (5.5) for  $\rho > 1$ . For this range, (5.9) shows that (5.5) reduces to

$$(5.13) \quad \int_1^{\infty} \phi(\xi) \int_0^{\infty} x^{\nu - s + 1} J_{\nu}(\xi x) J_s(\rho x) dx d\xi = \int_0^{\infty} x^{\nu - s + 1} J_s(\rho x) f(x) dx$$

and (5.6) shows that (5.13) becomes

$$(5.14) \quad \int_1^{\rho} (\rho^2 - \xi^2)^{\lambda - \nu - 1} \xi^{\nu} \phi(\xi) d\xi \\ = \rho^{\lambda} 2^{\lambda - \nu - 1} \Gamma(\lambda - \nu) \int_0^{\infty} x^{\nu - \lambda + 1} J_{\lambda}(\rho x) f(x) dx$$

for  $\rho > 1$ . This equation can also be reduced to a generalized Abel equation as the substitutions  $\xi^2 = z + 1$ ;  $\rho^2 = \sigma + 1$  show.

If the condition (5.4) is not satisfied and we have



$$(5.15) \quad x^\omega \int_0^1 J_\mu(\xi x) \phi(\xi) d\xi + \int_1^\infty J_\nu(\xi x) \phi(\xi) d\xi = f(x) \quad 0 < x < \infty$$

$$\mu \leq \nu + |\omega|$$

the solution of (5.15) is less easy to find. In this case it seems that we need to use preliminary differentiation and or integration techniques of the kind described in section 3 before we can reduce (5.15) to the solution of a pair of Abel equations. It is perhaps sufficient to note specifically that under appropriate assumptions a preliminary manipulation involving integration by parts can be used to reduce (5.15) to the case we have already considered.

As a simple formal illustration consider the equation

$$\int_0^1 \psi(\xi) \cos \xi x d\xi + \int_1^\infty \psi(\xi) \sin \xi x d\xi = \sqrt{x} f(x)$$

or the equivalent equation

$$(5.16) \quad \int_0^1 J_{-1/2}(\xi x) \phi(\xi) d\xi + \int_1^\infty J_{1/2}(\xi x) \phi(\xi) d\xi = f(x) \quad 0 < x < \infty$$

for which (5.4) is not satisfied. This equation can be expressed as

$$\begin{aligned} \int_0^1 \xi^{1/2} J_{-1/2}(x\xi) \frac{d}{d\xi} \int_0^\xi \zeta^{-1/2} \phi(\zeta) d\zeta d\xi \\ - \int_1^\infty \xi^{1/2} J_{1/2}(x\xi) \frac{d}{d\xi} \int_\xi^\infty \zeta^{-1/2} \phi(\zeta) d\zeta d\xi = f(x) . \end{aligned}$$

Integration by parts gives

$$f(x) = \int_0^x g(t) dt$$

THEOREM 1

Let  $f$  be a function defined on the interval  $[a, b]$ . If  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ . Moreover, if  $F$  is an antiderivative of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$f(x) = \frac{1}{x^2}$$

EXAMPLE 1

$$\int_1^2 \frac{1}{x^2} dx = \left[ -\frac{1}{x} \right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$$

THEOREM 2 (Fundamental Theorem of Calculus, Part II)

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

EXAMPLE 2



$$\begin{aligned}
(5.17) \quad & \int_0^1 J_{1/2}(x\xi) \xi^{1/2} \int_0^\xi \xi^{-1/2} \phi(\xi) d\xi d\xi \\
& + \int_1^\infty J_{-1/2}(x\xi) \xi^{1/2} \int_\xi^\infty \xi^{-1/2} \phi(\xi) d\xi d\xi \\
& = \frac{1}{x} \left[ f(x) - k_1 J_{-1/2}(x) - k_2 J_{1/2}(x) \right] \equiv F(x)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \int_0^1 \xi^{-1/2} \phi(\xi) d\xi \\
k_2 &= \int_1^\infty \xi^{-1/2} \phi(\xi) d\xi .
\end{aligned}$$

Equation (5.17) is now of the form (5.3) and it satisfies the condition (5.4). From (5.12) we find

$$\int_\rho^1 (\xi^2 - \rho^2)^{-1/2} \int_0^\xi \xi^{-1/2} \phi(\xi) d\xi d\xi = \sqrt{\frac{\pi}{2}} \int_0^\infty x^{1/2} J_0(\rho x) F(x) dx \quad 0 < \rho < 1$$

whose solution is

$$\begin{aligned}
(5.18) \quad & \int_0^\xi \xi^{-1/2} \phi(\xi) d\xi = -\sqrt{\frac{2}{\pi}} \frac{d}{d\xi} \int_\xi^1 \frac{\rho}{(\rho^2 - \xi^2)^{1/2}} \int_0^\infty \frac{J_0(\rho x)}{\sqrt{x}} \\
& \cdot \left[ f(x) - k_1 J_{-1/2}(x) - k_2 J_{1/2}(x) \right] dx d\rho .
\end{aligned}$$

From (5.14) we find that for  $\rho > 1$

$$\int_1^\rho (\rho^2 - \xi^2)^{-1/2} \int_\xi^\infty \xi^{-1/2} \phi(\xi) d\xi d\xi = \sqrt{\frac{\pi}{2}} \int_0^\infty x^{1/2} J_0(\rho x) F(x) dx$$

and the solution of this is



$$\begin{aligned}
 (5.19) \quad \int_{\xi}^{\infty} \xi^{-1/2} \phi(\xi) d\xi &= \sqrt{\frac{2}{\pi}} \frac{d}{d\xi} \int_1^{\xi} \frac{\rho}{\sqrt{\xi^2 - \rho^2}} \int_0^{\infty} \frac{J_0(\rho x)}{\sqrt{x}} \\
 &\quad \cdot \left[ f(x) - k_1 J_{-1/2}(x) - k_2 J_{1/2}(x) \right] dx d\rho .
 \end{aligned}$$

If appropriate conditions are imposed for  $f(x)$ , the constants  $k_1$  and  $k_2$  can be determined from (5.18) and (5.19) by letting  $\xi$  approach one from below and above.



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Institute of Mathematical  
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25 Waverly Place  
New York 3, N. Y. (1)

Professor G. Kuerti  
Department of Mechanical  
Engineering  
Case Institute of Technology  
Cleveland, Ohio (1)



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| Chapter LXXXXIII  | 930  |
| Chapter LXXXXIV   | 940  |
| Chapter LXXXXV    | 950  |
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Burt Hochman  
A. Kadet

N. Y. U. Institute of  
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